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EXPLICIT BLOCK DIAGONAL DECOMPOSITION OF BLOCK MATRICES CORRESPONDING TO SYMMETRIC AND REGULAR STRUCTURES OF FINITE SIZE

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ABSTRACT

Some sufficient conditions for the explicit block diagonal decomposition of block matrices are developed. These applied to matrices corresponding to symmetric and regular structures of finite size. A special numerical procedure is proposed for solving linear systems with quasi block Toeplitz matrix. An explicit formula for the natural frequencies of a clamped rectangular plate is derived.

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1. INTRODUCTION

1. Large scale matrices (we consider them as block matrices A_*) usually correspond to physical structures (models) which contain identical or periodically repeated substructures (submodels), for example, matrices corresponding to symmetric or regular structures (models). In some cases these matrices can be explicitly block diagonalized or even diagonalized. They are the subject of this paper.

2. Introduce some definitions. Let A be n×n non-defective matrix (i.e., it has a full set of eigenvectors)

$$A = [a_{ij}]_{i,j=1}^{n} \tag{1.1}$$

The following factorization of A is called the spectral decomposition

$$A = U \Lambda U^{-1} \tag{1.2}$$

Here U is the modal matrix (its columns are eigenvectors of A) and Λ is the spectral matrix (diagonal matrix with eigenvalues of A),

$$U = [u_{ij}]_{i,j=1}^{n} , \qquad \Lambda = \lceil \lambda_j \rfloor_{j=1}^{n} \tag{1.3}$$

Special brackets $\lceil \ldots \rfloor$ and one subscript are used for diagonal and block diagonal matrices: $\lceil \lambda_j \rfloor_{j=1}^{n} \equiv [\delta_{ij}\lambda_j]_{i,j=1}^{n}$, δ_{ij} is the Kronecker delta. We shall call (1.2) the explicit spectral decomposition of A (1.1) if λ_j and u_{ij} are given as explicit functions in a_{ij} and n. Let A_* be a block matrix of order mn with m×m blocks

$$A_* = [A_{ij}]_{i,j=1}^{n} = [[a_{ij\sigma\tau}]_{\sigma,\tau=1}^{m}]_{i,j=1}^{n} \tag{1.4}$$

and $\tilde{A}_*$ be another block matrix with the same elements $a_{ij\sigma\tau}$ grouped in $n\times n$ blocks

$$\tilde{A}_* = [\tilde{A}_{\sigma\tau}]_{\sigma,\tau=1}^{m} = [[a_{ij\sigma\tau}]_{i,j=1}^{n}]_{\sigma,\tau=1}^{m} \qquad (1.5)$$

Both matrices are similar

$$\tilde{A}_* = P_*^T A_* P_* \ , \qquad (1.6)$$

where P_* is the following permutation matrix with rectangular $m\times n$ blocks

$$P_* = [[\delta_{ij}\delta_{\sigma\tau}]_{\sigma,j=1}^{m,n}]_{i,\tau=1}^{n,m} \qquad (1.7)$$

Obviously

$$P_*^{-1} = P_*^T = [[\delta_{ij}\delta_{\sigma\tau}]_{j,\sigma=1}^{n,m}]_{\tau,i=1}^{m,n} \qquad (1.8)$$

The following block diagonal decomposition of A_* (1.4)

$$A_* = U_* \Lambda_* U_*^{-1} \ , \qquad (1.9)$$

where Λ_* is block diagonal, is called the explicit block diagonal decomposition of A_* if blocks of Λ_* and U_* are explicit functions in eigenvalues and eigenvectors of A_{ij} and m,n.

3. Great advantage in solving

$$(A_* - \lambda I)x_* = b_* \qquad (1.10)$$

can be achieved if A_* possesses the explicit block diagonal decomposition. The necessary condition for the explicit block diagonal decomposition of A_* (1.4) are unknown, probably, do not exist. However, some sufficient conditions can be easily established. They are given in Section 2. In Section 3 we apply them to matrices corresponding to symmetric structures of finite size. Matrices associated with regular structures of finite size are quasi block Toeplitz matrices. In some particular cases they can be explicitly block diagonalized, Section 4. We propose in Section 5 a special numerical method for solving system (1.10) with quasi block Toeplitz matrix A_* in general case, Fast Block Elimination. Finite difference block matrices corresponding to the biharmonic operator over a rectangular region are considered in Section 6. The asymptotic explicit spectral decomposition of such matrices is constructed. It leads to the explicit formula for natural frequencies of a clamped rectangular plate which contains the Navier formula for a simply supported plate, as a particular case.

4. All proposed decompositions may be viewed as an extension of the Poisson's solver [1,2]. Fast Block Elimination, which is considered in Section 5, is based on an idea of simultaneous elimination of all even subvectors $x_2, x_4, \ldots$ of (1.10) and taken from the Fast Fourier Transform. It was also utilized in [1], the "cyclic odd-even reduction and factorization (CORF)". However, such an idea, being applied alone, is not effective. As said in [1], "from computational viewpoint, the CORF algorithm, as developed here, is virtually useless." We combine it with a special numerical procedure for computation of a new system at each step. This makes Fast Block

Elimination more effective. Its efficiency is compared with the One-Way Dissection Method [3]. It is found that the proposed method is more effective when $\alpha = n/m \gtrsim 1.3$ (see Section 5).

2. SUFFICIENT CONDITIONS FOR EXPLICIT BLOCK DIAGONAL DECOMPOSITION OF MATRIX A_*

1. <u>THEOREM 2.1</u>:

If all blocks A_{ij} of A_* commute, then there exists the explicit block diagonal decomposition of A_* (1.4).

<u>PROOF</u>: Suppose all A_{ij}, $i,j=1,\ldots,n$ are commutative matrices: $[A_{ij}, A_{i_1 j_1}] = 0_m$, $i,i_1,j,j_1 = 1,\ldots,n$. Then all A_{ij} have the same modal matrix V, i.e.,

$$A_{ij} = VN_{ij}V^{-1}, \qquad i,j=1,\ldots,n \tag{2.1}$$

where

$$N_{ij} = \lceil \nu_{ij\tau} \rfloor_{\tau=1}^{m}, \qquad V = [v_{\sigma\tau}]_{\sigma,\tau=1}^{m}, \qquad V^{-1} = [\tilde{v}_{\sigma\tau}]_{\sigma,\tau=1}^{m} \tag{2.2}$$

Thus

$$A_* = [VN_{ij}V^{-1}]_{i,j=1}^{n} = (I_n \times V)N_*(I_n \times V^{-1}),$$

where N_* is a full block matrix, but its blocks are diagonal

$$N_* = [N_{ij}]_{i,j=1}^{n} = [\lceil \nu_{ij\tau} \rfloor_{\tau=1}^{m}]_{i,j=1}^{n}$$

and "x" is the Kronecker (tensor) multiplication. Therefore one can write

$$A_* = [(I_n \times V)P_*]\,[P_*^T N_* P_*]\,[P_*(I_n \times V^{-1})] = X_* \tilde{N}_* X_*^{-1} \tag{2.3}$$

where $\tilde{N}_*$ is block diagonal with blocks of order n

$$\tilde{N}_* = \lceil \tilde{N}_\tau \rfloor_{\tau=1}^{m} = \lceil [\nu_{ij\tau}]_{i,j=1}^{n} \rfloor_{\tau=1}^{m} \tag{2.4}$$

and X_* is a full block matrix with rectangular m×n blocks

$$X_* = (I_n \times V)P_* = [[\delta_{ij} v_{\sigma\tau}]_{\sigma,j=1}^{m,n}]_{i,\tau=1}^{n,m} , \tag{2.5}$$

$$X_*^{-1} = P_*^T(I_n \times V^{-1}) = [[\delta_{ij} \tilde{v}_{\sigma\tau}]_{i,\tau=1}^{n,m}]_{\sigma,j=1}^{m,n} , \tag{2.6}$$

<u>THEOREM</u> <u>2.2</u>: If all blocks $\tilde{A}_{\sigma\tau}$ of $\tilde{A}_*$ (1.5) commute and therefore they have the following explicit spectral decomposition

$$\tilde{A}_{\sigma\tau} = \tilde{U}\tilde{M}_{\sigma\tau}\tilde{U}^{-1} , \qquad \sigma,\tau=1,\ldots,m \tag{2.7}$$

where

$$\tilde{M}_{\sigma\tau} = \lceil \mu_{j\sigma\tau} \rfloor_{j=1}^{n} , \quad \tilde{U} = [u_{ij}]_{i,j=1}^{n} , \quad \tilde{U}^{-1} = [\tilde{u}_{ij}]_{i,j=1}^{n} , \tag{2.8}$$

then matrix A_* (1.4) has the following explicit block diagonal decomposition:

$$A_* = U_* M_* U_*^{-1} , \tag{2.9}$$

with

$$M_* = \lceil M_j \rfloor_{j=1}^{n} = \lceil [\mu_{j\sigma\tau}]_{\sigma,\tau=1}^{m} \rfloor_{j=1}^{n} , \tag{2.10}$$

$$U_* = \tilde{U}\times I_m = [u_{ij}I_m]_{i,j=1}^n \qquad (2.11)$$
$$U_*^{-1} = \tilde{U}^{-1}\times I_m = [\tilde{u}_{ij}I_m]_{i,j=1}^n$$

PROOF: According to Theorem 2.1

$$\tilde{A}_* = Y_* M_* Y_*^{-1} , \qquad (2.12)$$

where M_* is of the form (2.10) and

$$Y_* = (I_m\times\tilde{U})P_*^T , \qquad Y_*^{-1} = (I_m\times\tilde{U}^{-1})P_* \qquad (2.13)$$

Therefore

$$A_* = P_*\tilde{A}_*P_*^T = (P_*Y_*)M_*(Y_*^{-1}P_*^T) = U_*M_*U_*^{-1}$$

with

$$U_* = P_*(I_m\times\tilde{U})P_*^T = \tilde{U}\times I_m$$
$$U_*^{-1} = P_*(I_m\times\tilde{U}^{-1})P_*^T = \tilde{U}^{-1}\times I_m$$

Note that in (2.9) matrices U_* and M_* have blocks of the same order m as matrix A_*.

2. THEOREM 2.3:

If both matrices A_* (1.4) and $\tilde{A}_*$ (1.5) contain commutative blocks then A_* has the explicit spectral decomposition

$$\Lambda_* = W_* \, \Lambda_* \, W_*^{-1} , \tag{2.14}$$

where Λ_* is the spectral matrix

$$\Lambda_* = \left[\left[\lambda_{j\tau} \right]_{\tau=1}^{m} \right]_{j=1}^{n} , \tag{2.15}$$

$$\lambda_{j\tau} = \sum_{s,t=1}^{n} \nu_{st\tau} \tilde{u}_{js} u_{tj} = \sum_{\alpha,\beta=1}^{m} \mu_{j\alpha\beta} \tilde{v}_{\tau\alpha} v_{\beta\tau} \tag{2.16}$$

and W_* is the modal matrix

$$\begin{aligned} W_* &= \tilde{U} \times V = [[u_{ij} v_{\sigma\tau}]_{\sigma,\tau=1}^{m}]_{i,j=1}^{n} \\ W_*^{-1} &= \tilde{U}^{-1} \times V^{-1} = [[\tilde{u}_{ij} \tilde{v}_{\sigma\tau}]_{\sigma,\tau=1}^{m}]_{i,j=1}^{n} \end{aligned} \tag{2.17}$$

To prove this theorem, note that matrix A_* now possesses two block diagonal decompositions given by (2.3) and (2.9). It follows from here that

$$\begin{aligned} M_* &= (\tilde{U}^{-1} \times I_m)(I_n \times V) N_* (I_n \times V^{-1})(\tilde{U} \times I_m) \\ &= (\tilde{U}^{-1} \times V) N_* (\tilde{U} \times V^{-1}) = \left[V \left(\sum_{s,t=1}^{n} N_{st} \tilde{u}_{is} u_{tj} \right) V^{-1} \right]_{i,j=1}^{n} \\ &= (I_n \times V) \left[\sum_{s,t=1}^{n} N_{st} \tilde{u}_{is} u_{tj} \right]_{i,j=1}^{n} (I_n \times V^{-1}) \\ &= (I_n \times V) \Lambda_* (I_n \times V^{-1}) \end{aligned} \tag{2.18}$$

Matrix M_* (2.10) is block diagonal, hence, all off-diagonal blocks of Λ_* are zero submatrices

$$\sum_{s,t=1}^{n} N_{st}\tilde{u}_{is}u_{tj} = 0_m \ , \qquad t \neq s$$

Since all N_{st} (2.2) are diagonal, so are the diagonal blocks of Λ_*

$$\Lambda_j = \sum_{s,t=1}^{n} N_{st}\tilde{u}_{is}u_{tj} = \lceil \lambda_{j\tau} \rfloor_{\tau=1}^{m} \ , \qquad j=1,\ldots,n$$

i.e.,

$$\lambda_{j\tau} = \sum_{s,t=1}^{n} \nu_{st\tau}\tilde{u}_{is}u_{tj} \ , \qquad j=1,\ldots,n \ , \quad \tau=1,\ldots,m$$

Thus,

$$A_* = (\tilde{U}\times I_m)M_*(\tilde{U}^{-1}\times I_m) = (\tilde{U}\times V)\Lambda_*(\tilde{U}^{-1}\times V^{-1})$$

Finally we note that $\tilde{A}_*$ (1.5) also has the explicit spectral decomposition

$$\tilde{A}_* = \tilde{W}_*\tilde{\Lambda}_*\tilde{W}^{-1} \ , \tag{2.19}$$

where

$$\tilde{\Lambda}_* = \lceil \tilde{\Lambda}_\tau \rfloor_{\tau=1}^{m} = \lceil \lceil \lambda_{j\tau} \rfloor_{j=1}^{n} \rfloor_{\tau=1}^{m} \ , \tag{2.20}$$

$$\lambda_{j\tau} = \sum_{\alpha,\beta=1}^{m} \mu_{j\alpha\beta}\tilde{v}_{\tau\alpha}v_{\beta\tau}$$

$$\tilde{W}_* = V\times\tilde{U} \ , \qquad \tilde{W}_* = V^{-1}\times\tilde{U}^{-1} \tag{2.21}$$

3. Consider one particular case of matrix A_* (1.4) which is the basis for next sections:

$$A_* = \sum_{r=1}^{p} B_r \times C_r \ , \tag{2.22}$$

where B_r and C_r, $r=1,\dots,p$, are $n\times n$ and $m\times m$ matrices, respectively. Since

$$\tilde{A}_* = P_*^T\Big(\sum_{r=1}^{p} B_r \times C_r\Big)P_* = \sum_{r=1}^{p} C_r \times B_r \tag{2.23}$$

we have

$$A_{ij} = \sum_{r=1}^{p} b_{ij}^{(r)} C_r \quad \text{and} \quad \tilde{A}_{\sigma\tau} = \sum_{r=1}^{p} c_{\sigma\tau}^{(r)} B_r \qquad i,j=1,\dots,n; \quad \sigma,\tau=1,\dots,m \tag{2.24}$$

Thus, one can formulate the following statements:

<u>THEOREM</u> <u>2.4</u>:

(i) If

$$B_r = \tilde{U}\tilde{M}_r\tilde{U}^{-1} \ , \quad \tilde{M}_r = \lceil \mu_j^{(r)} \rfloor_{j=1}^{n} \ , \quad \tilde{U} = [u_{ij}]_{i,j=1}^{n} \ , \quad r=1,\dots,p \tag{2.25}$$

then there is the explicit block diagonal decomposition of A_* (2.22)

$$A_* = U_* M_* U_*^{-1} \ , \tag{2.26}$$

where U_* is of the form (2.11) and

$$M_* = \sum_{r=1}^{p} M_r \times C_r = \lceil \sum_{r=1}^{p} \mu_j^{(r)} C_r \rfloor_{j=1}^{n} \qquad (2.27)$$

(ii) If

$$C_r = VN_rV^{-1} \ , \quad N_r = \lceil \nu_\tau^{(r)} \rfloor_{\tau=1}^{n} \ , \quad V = [v_{\sigma\tau}]_{\sigma,\tau=1}^{m} \ , \qquad r=1,\ldots,p \qquad (2.28)$$

then

$$A_* = X_* N_* X_*^{-1} \ , \qquad (2.29)$$

where X_* is of the form (2.5) and

$$\bar{N}_* = \sum_{r=1}^{p} N_r \times B_r = \lceil \sum_{r=1}^{p} \nu_\tau^{(r)} B_r \rfloor_{\tau=1}^{m} \qquad (2.30)$$

(iii) If conditions (i) and (ii) are satisfied simultaneously, then A_* (2.22) possesses the explicit spectral decomposition

$$A_* = W_* \Lambda_* W_*^{-1} \ , \qquad (2.31)$$

where W_* is of the form (2.17) and

$$\Lambda_* = \lceil \lceil \sum_{r=1}^{p} \mu_j^{(r)} \nu_\tau^{(r)} \rfloor_{\tau=1}^{m} \rfloor_{j=1}^{n} \qquad (2.32)$$

<u>REMARK</u>: Last statement is well-known [4].

3. BLOCK MATRICES CORRESPONDING TO SYMMETRIC STRUCTURES OF FINITE SIZE

1. We define symmetric structure (model) of finite size, S, as a physical system which possesses a point-symmetry group G of order $n > 1$.* Clearly each such structure contains n identical elementary regions (or cells) $S_1, S_2, \ldots, S_n$. We call S_1 the fundamental region. Suppose it has m degrees of freedom, then matrix A_* corresponding to S is a block matrix of order mn and can be presented by (1.4).

2. Constructing matrix A_* we shall obey the following Symmetry Rule which states that we are free to choose all variables and coordinate system for the fundamental region only. The variables and coordinate system corresponding to region S_j should be obtained by applying the symmetry transformation $g_j \in G$, $j=1,\ldots,n$, to the fundamental regions S_1.

3. Let g_i, g_j be elements of the group G and

$$g_{w(i,j)} = g_i g_j \,, \qquad i,j=1,\ldots,n \tag{3.1}$$

Introduce the following n permutation matrices of order n

$$Q(g_j) \equiv [q_{ik}(g_j)]_{i,k=1}^{n} = [\delta_{k,w(i,j)}]_{i,k=1}^{n} \,, \qquad j=1,\ldots,n \tag{3.2}$$

which are in one-to-one correspondence with the group table. Denote by τ_r ($r=1,\ldots,H$) the r^{th} irreducible representation of group G. Its matrices are $n_r \times n_r$ unitary matrices (we consider finite groups)

---

*There are fourteen point-symmetry groups, and, hence, fourteen different types of symmetric structures of finite size.

$$\tau_r(g_j) = [\tau_{r\alpha\beta}(g_j)]_{\alpha,\beta=1}^{n_r} , \qquad r=1,\dots,H;\ j=1,\dots,n \tag{3.3}$$

For point-symmetry groups $1 \leq n_r \leq 5$. Elements $\tau_{r\alpha\beta}(g_j)$ of all irreducible representations satisfy some orthogonality relations, in particular,

$$\sum_{r=1}^{H} \frac{n_r}{n} \sum_{\alpha,\beta=1}^{n_r} \tau_{r\alpha\beta}(g_i)\bar{\tau}_{r\alpha\beta}(g_k) = \delta_{ik} , \qquad i,k=1,\dots,n \tag{3.4}$$

<u>LEMMA</u> <u>3.1</u>: There exists the explicit block diagonal decomposition of matrices $Q(g_j)$ (3.2)

$$Q(g_j) = UT(g_j)U^H , \qquad j=1,\dots,n \tag{3.5}$$

where $T(g_j)$ are block diagonal unitary matrices having the same configuration for all $g_j \in G$, and U is a uninormal matrix, $U^H \equiv \bar{U}^T$, namely,

$$T(g_j) = \begin{bmatrix} \underbrace{\begin{matrix}\tau_1(g_j) & & \\ & \cdots & \\ & & \tau_1(g_j)\end{matrix}}_{n_1 \text{ TIMES}} & & & \\ & \underbrace{\begin{matrix}\tau_2(g_j) & & \\ & \cdots & \\ & & \tau_2(g_j)\end{matrix}}_{n_2 \text{ TIMES}} & & \\ & & \ddots & \\ & & & \underbrace{\begin{matrix}\tau_H(g_j) & & \\ & \cdots & \\ & & \tau_H(g_j)\end{matrix}}_{n_H \text{ TIMES}} \end{bmatrix} \tag{3.6}$$

or in short

$$T(g_j) = \lceil \lceil \delta_{\gamma\gamma}\tau_r(g_j)\rfloor_{\gamma=1}^{n_r} \rfloor_{r=1}^{H}$$

$$= \lceil \lceil [\delta_{\gamma\gamma}\tau_{r\alpha\beta}(g_j)]_{\alpha,\beta=1}^{n_r} \rfloor_{\gamma=1}^{n_r} \rfloor_{r=1}^{H} \quad , \quad j=1,\ldots,n \tag{3.7}$$

and

$$U = [u_{is}]_{i,s=1}^{n} = [(\frac{n_r}{n})^{1/2}[[\tau_{r\gamma\alpha}(g_i)]_{\alpha=1}^{n_r}\ ]_{\gamma=1}^{n_r}\ ]_{i,r=1}^{n,H} \tag{3.8}$$

Here column subscript, s, is associated with three subscripts r, α and γ. Since $\sum_{r=1}^{H} n_r^2 = n$, subscript s runs from 1 to n when α and γ run from 1 to n_r and r runs from 1 to H.

$$U^H = [\bar{u}_{sk}]_{s,k=1}^{n} = [(\frac{n_r}{n})^{1/2}[[\bar{\tau}_{r\gamma\beta}(g_k)]_{\beta=1}^{n_r}\ ]_{\gamma=1}^{n_r}]_{r,k=1}^{H,n} \tag{3.9}$$

<u>PROOF</u>: Equation (3.5) is a matrix form of the following orthogonality relation

$$\sum_{r=1}^{H} \frac{n_r}{n} \sum_{\alpha,\beta,\gamma=1}^{n_r} \tau_{r\gamma\alpha}(g_i)\tau_{r\alpha\beta}(g_j)\bar{\tau}_{r\gamma\beta}(g_k) = \delta_{k,w(i,j)} \ , \quad i,j,k=1,\ldots,n \tag{3.10}$$

which generalizes (3.4).

□

4. <u>THEOREM</u> <u>3.1</u>:

Block matrix A_* (1.4), associated with symmetric structure S and

constructed in accordance with the symmetry rule, can be expressed by the following Structural Formula:

$$A_* = \sum_{j=1}^{n} Q(g_j) \times A_{1j} , \qquad (3.11)$$

where A_{1j} are blocks of the first block row of A_*.

<u>PROOF</u>: Introduce the identical symmetric structure $S' \equiv S$ with elementary regions $S_1' = S_j$, $S_2' = S_{j+1}, \ldots$. Suppose that we coincide S' and S by a certain symmetry transformation $g \in G$ so that S_1' be coincided with S_i. Then S_j' will coincide with $S_{w(i,j)}$, where $w(i,j)$ is defined by (3.1). Suppose matrix $A_* = [A_{pj}]_{p,j=1}^n$ corresponds to S while $A_*' = [A_{pj}']_{p,j=1}^n$ to S'. Clearly

$$A_{pj}' = A_{w(i,p),w(i,j)} , \qquad i,j,p=1,\ldots,n$$

On the other hand, S' and S are identical, thus, their matrices while satisfying the symmetry rule, have to be equal. Hence $A_{pj}' = A_{pj}$. Therefore

$$A_{pj} = A_{w(i,p),w(i,j)} , \qquad i,j,p=1,\ldots,n$$

Let p be the equal to one. Then, since $w(i,1) = i$, (3.1), $A_{1j} = A_{i,w(i,j)}$ or

$$A_{1j} = \sum_{\ell=1}^{n} A_{i\ell}\delta_{\ell,w(i,j)} , \qquad j=1,\ldots,n$$

Multiplying both sides by $\delta_{k,w(i,j)}$ and summing up with respect to j from 1 to n, we obtain

$$\sum_{j=1}^{n} A_{ij}\delta_{k,w(i,j)} = A_{ik} , \qquad i,k=1,\ldots,n$$

or, by virtue of (3.2)

$$A_{ik} = \sum_{j=1}^{n} q_{ik}(g_j)A_{1j} , \qquad i,k=1,\ldots,n \tag{3.12}$$

the Structural Formula (3.11) follows from here.

□

5. <u>THEOREM 3.2</u>:

Block matrix A_* (3.11) has the following explicit block diagonal decomposition:

$$A_* = U_*\Lambda_*U_*^H , \tag{3.13}$$

where U_* is uninormal

$$U_* = U\times I_m = [u_{is}I_m]_{i,s=1}^{n} \tag{3.14}$$

and Λ_* is Hermitian, block diagonal

$$\Lambda_* = \sum_{j=1}^{n} T(g_j)\times A_{1j} = [\ [\ \delta_{\gamma\gamma}\Lambda_r\ ]_{\gamma=1}^{n_r}\ ]_{r=1}^{H} \tag{3.15}$$

with blocks Λ_r of order mn_r, $1 \leq n_r \leq 5$

$$\Lambda_r = \sum_{j=1}^{n} \tau_r(g_j)\times A_{1j} = [\sum_{j=1}^{n} \tau_{r\alpha\beta}(g_j)A_{1j}]_{\alpha,\beta=1}^{n_r} \tag{3.16}$$

PROOF: Equations (3.13)-(3.16) follow from (3.11) if we substitute there (3.5)-(3.9).

□

NOTE: Symmetric structure S has at least n_r-fold natural frequencies and critical loads, $1 \leq n_r \leq 5$, because, as it follows from (3.15), matrix A_* has at least n_r-fold eigenvalues.

THEOREM 3.3: Block matrix A_*^{-1} can be presented in the form

$$A_*^{-1} = \sum_{j=1}^{n} Q(g_j)\times\tilde{A}_{1j} , \tag{3.17}$$

where

$$\tilde{A}_{1j} = \sum_{r=1}^{H} \frac{n_r}{n} \sum_{\alpha,\beta=1}^{n_r} \tilde{\Lambda}_{r\alpha\beta}\bar{\tau}_{r\alpha\beta}(g_j) , \qquad j=1,...,n \tag{3.18}$$

and $\tilde{\Lambda}_{r\alpha\beta}$ are blocks of

$$\tilde{\Lambda}_r^{-1} = [\tilde{\Lambda}_{r\alpha\beta}]_{\alpha,\beta=1}^{n_r} \tag{3.19}$$

PROOF: We find from (3.11) that

$$A_*^{-1} = \sum_{j=1}^{n} Q(g_j)\times\tilde{A}_{1j} ,$$

where $\tilde{A}_{1j}$ are blocks of the first block row of $\tilde{A}_*$. On the other hand

$$A_*^{-1} = U_* \Lambda_*^{-1} U_*^H$$

and therefore

$$\Lambda_*^{-1} = \sum_{j=1}^{n} T(g_j)\times\tilde{A}_{1j} = \left[\left[\delta_{\gamma\gamma}\Lambda_r^{-1} \right]_{\gamma=1}^{n_r} \right]_{r=1}^{H}$$

Suppose blocks Λ_r^{-1}, r=1,...,H are computed. Then according to (3.16)

$$\tilde{\Lambda}_{r\alpha\beta} = \sum_{j=1}^{n} \tau_{r\alpha\beta}(g_j)\tilde{A}_{1j} , \qquad \alpha,\beta=1,\dots,n_r \ ; \quad r=1,\dots,H$$

Multiplying both sides by $(n_r/n)\bar{\tau}_{r\alpha\beta}(g_k)$ and summing up with respect to α and β from 1 to n_r and r from 1 to H, we obtain, taking into account the orthogonality relation (3.4),

$$\sum_{r=1}^{H} \frac{n_r}{n} \sum_{\alpha,\beta=1}^{n_r} \tilde{\Lambda}_{r\alpha\beta}\bar{\tau}_{r\alpha\beta}(g_k)$$

$$= \sum_{j=1}^{n} \tilde{A}_{1j} \sum_{r=1}^{H} \frac{n_r}{n} \sum_{\alpha,\beta=1}^{n_r} \tau_{r\alpha\beta}(g_j)\bar{\tau}_{r\alpha\beta}(g_k) = \sum_{j=1}^{n} \tilde{A}_{1j}\delta_{jk} = \tilde{A}_{1k}$$

Thus, to invert mn×mn matrix A_* (3.11) we need to invert only the matrices Λ_r (r=1,...,H) of order mn_r, $1 \le n_r \le 5$. Note that if A_* is symmetric, some blocks $\tilde{A}_{1j}$ are mutually transposed, we have to compute only truly different $\tilde{A}_{1j}$.

6. THEOREM 3.4: Linear system

$$A_* x_* = b_* \tag{3.20}$$

of order mn is split into H decoupled subsystems of order mn_r, $1 \leq n_r \leq 5$, containing n_r unknown subvectors $y_{r\gamma}$, $\gamma=1,\ldots,n_r$, each

$$\Lambda_r [y_{r1},\ldots,y_{rn_r}] = [c_{r1},\ldots,c_{rn_r}] , \qquad r=1,\ldots,H \tag{3.21}$$

where

$$c_{r\gamma} = \left(\frac{n_r}{n}\right)^{1/2} \left[\sum_{k=1}^{n} b_k \bar{\tau}_{r\gamma\beta}(g_k) \right]_{\beta=1}^{n_r} \qquad \gamma=1,\ldots,n_r;\ r=1,\ldots,H \tag{3.22}$$

Once (3.21) are solved, the original unknowns $x_* = [x_i]_{i=1}^{n}$ are determined by

$$x_i = \sum_{r=1}^{H} \left(\frac{n_r}{n}\right)^{1/2} \sum_{\gamma,\alpha=1}^{n_r} y_{r\gamma\alpha} \tau_{r\gamma\alpha}(g_i) , \qquad i=1,\ldots,n \tag{3.23}$$

where $y_{r\gamma\alpha}$ are subvectors of $y_{r\gamma} = [y_{r\gamma\alpha}]_{\alpha=1}^{n_r}$. In fact, substituting (3.13) into (3.20) and, taking into account (3.8)-(3.9), we obtain (3.21)-(3.23).

7. Finally, it is necessary to note that the Structural Formula (3.11) can significantly simplify the use of tetrahedrons and cubes as finite elements because the corresponding matrix A_* is symmetric and therefore it is sufficient to compute and store only one-half of its first block row.

4. BLOCK MATRICES CORRESPONDING TO REGULAR STRUCTURE OF FINITE SIZE

1. We define the regular structure (model) of finite size as a structure (model) which is formed by a finite number of identical (or periodically repeated) substructures (submodels) of finite size. Matrices associated with regular structures are quasi block Toeplitz matrices. Their blocks at the left top and right bottom corners express the boundary conditions and therefore they usually distinguish from others in the same block codiagonal.

2. Introduce the following auxiliary quasi Toeplitz matrix of order n

$$B_1 = \begin{bmatrix} \boxed{-f} & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \cdot & \cdot & \cdot & \cdot \\ & & & 1 & 0 & 1 \\ & & & & 1 & \boxed{-g} \end{bmatrix}_n , \quad |f|, |g| \le 1 \tag{4.1}$$

<u>LEMMA</u> <u>4.1</u>: There is the explicit spectral decomposition of B_1

$$B_1 = UM_1U^T , \tag{4.2}$$

where

$$M_1 = \lceil \mu_j^{(1)} \rfloor_{j=1}^{n} , \qquad \mu_j^{(1)} = 2 \cos \theta_j \tag{4.3}$$

$$U = [u_{ij}]_{i,j=1}^{n} \tag{4.4}$$

$$\begin{aligned} u_{ij} &= c_j\left(\sin i\theta_j + f \sin (i-1)\theta_j\right) \\ &= c_j'\left(g \sin (n-i)\theta_j + \sin (n+1-i)\theta_j\right) \end{aligned} \tag{4.5}$$

Arguments θ_j, $j=1,\ldots,n$ are zeros of the characteristic polynomial

$$\Delta(\theta) = \sin (n+1)\theta + (f+g) \sin n\theta + fg \sin (n-1)\theta \tag{4.6}$$

There are n distinct zeros θ_j on $(0,\pi]$. We omit the proof because it is the same as for Lemma 6.1.

Introduce a Chebyshev family of matrices B

$$\begin{aligned} &B_o = I_n \\ &B_1 \\ &B_2 = B_1^2 - 2B_o \\ &B_3 = B_1^3 - 3B_1 \\ &B_4 = B_1^4 - 4B_1^2 + 2B_o \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{4.7}$$

They are quasi Toeplitz matrices and are given explicitly by (4.8) thru (4.11) for n=9.

$$B_1 = \begin{bmatrix} -f & 1 & & & & & & & \\ 1 & & 1 & & & & & & \\ & 1 & & 1 & & & & & \\ & & 1 & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & & & 1 & & 1 & & \\ & & & & & 1 & & 1 & \\ & & & & & & 1 & & 1 \\ & & & & & & & 1 & -g \end{bmatrix} \tag{4.8}$$

$$B_2 = \begin{bmatrix} -(1-f^2) & -f & 1 & & & & & & \\ -f & & & 1 & & & & & \\ 1 & & & & 1 & & & & \\ & 1 & & & & 1 & & & \\ & & 1 & & & & 1 & & \\ & & & 1 & & & & 1 & \\ & & & & 1 & & & & 1 \\ & & & & & 1 & & & -g \\ & & & & & & 1 & -g & -(1-g^2) \end{bmatrix} \tag{4.9}$$

$$B_3 = \begin{bmatrix} f(1-f^2) & -(1-f^2) & -f & 1 & & & & & \\ -(1-f^2) & -f & & & 1 & & & & \\ -f & & & & & 1 & & & \\ 1 & & & & & & 1 & & \\ & 1 & & & & & & 1 & \\ & & 1 & & & & & & 1 \\ & & & 1 & & & & & -g \\ & & & & 1 & & & -g & -(1-g^2) \\ & & & & & 1 & -g & -(1-g^2) & g(1-g^2) \end{bmatrix} \tag{4.10}$$

$$B_4 = \begin{bmatrix} -f^2(1-f^2) & f(1-f^2) & -(1-f^2) & -f & 1 & & & & \\ f(1-f^2) & -(1-f^2) & -f & & & 1 & & & \\ -(1-f^2) & -f & & & & & 1 & & \\ -f & & & & & & & 1 & \\ 1 & & & & & & & & 1 \\ & 1 & & & & & & & -g \\ & & 1 & & & & & -g & -(1-g^2) \\ & & & 1 & & & -g & -(1-g^2) & g(1-g^2) \\ & & & & 1 & -g & -(1-g^2) & g(1-g^2) & -g^2(1-g^2) \end{bmatrix} \tag{4.11}$$

Clearly,

$$B_r = UM_rU^T \tag{4.12}$$

with the eigenvalues

$$\mu_j^{(r)} = 2\cos r\theta_j\ , \qquad j=1,\ldots,n \tag{4.13}$$

Introduce also a family of non-symmetric matrices $\hat{B}_r$ of the same order n

$$\hat{B}_o = I_n\ , \qquad \hat{B}_1\ , \qquad \hat{B}_2 = \hat{B}_1^2 - 2\alpha\hat{B}_o\ ,\ldots \tag{4.14}$$

Their explicit forms are given by (4.15)-(4.16) for n=9

$$\hat{B}_1 = \begin{bmatrix} -f & 1 & & & & & & & \\ \alpha & & 1 & & & & & & \\ & \alpha & & 1 & & & & & \\ & & \alpha & & 1 & & & & \\ & & & \alpha & & 1 & & & \\ & & & & \alpha & & 1 & & \\ & & & & & \alpha & & 1 & \\ & & & & & & \alpha & & 1 \\ & & & & & & & \alpha & g \end{bmatrix} \tag{4.15}$$

$$\hat{B}_2 = \begin{bmatrix} -(\alpha-f^2) & -f & 1 & & & & & & \\ -\alpha f & & & 1 & & & & & \\ \alpha^2 & & & & 1 & & & & \\ & \alpha^2 & & & & 1 & & & \\ & & \alpha^2 & & & & 1 & & \\ & & & \alpha^2 & & & & 1 & \\ & & & & \alpha^2 & & & & 1 \\ & & & & & \alpha^2 & & & -g \\ & & & & & & \alpha^2 & -\alpha g & -(\alpha-g^2) \end{bmatrix} \tag{4.16}$$

<u>LEMMA 4.2</u>: There exists the explicit spectral decomposition of $\hat{B}_r$:

$$\hat{B}_r = \hat{U}\hat{M}_r\hat{U}^{-1} , \tag{4.17}$$

where

$$\hat{M}_r = \lceil \hat{\mu}_j^{(r)} \rfloor_{j=1}^{n} , \qquad \hat{\mu}_j^{(r)} = 2\alpha^{r/2} \cos r\theta_j , \tag{4.18}$$

$$\left.\begin{aligned} \hat{U} &= [\hat{u}_{ij}]_{i,j=1}^{n} , & \hat{u}_{ij} &= \alpha^{i/2} v_{ij} , \\ \hat{U}^{-1} &= [\hat{\hat{u}}_{ij}]_{i,j=1}^{n} , & \hat{\hat{u}}_{ij} &= \alpha^{-j/2} v_{ji} , \end{aligned}\right\} \tag{4.19}$$

This lemma immediately follows from Lemma 4.1 if we note that there is a diagonal matrix S

$$S = \lceil \alpha^{j/2} \rfloor_{j=1}^{n} \tag{4.20}$$

such that $S^{-1}\hat{B}_rS$ is symmetric quasi Toeplitz. For instance,

$$S^{-1}\hat{B}_1 S = \begin{bmatrix} \boxed{-f} & \sqrt{\alpha} & & & & \\ \sqrt{\alpha} & 0 & \sqrt{\alpha} & & & \\ & \sqrt{\alpha} & 0 & \sqrt{\alpha} & & \\ & & \cdots & \cdots & \cdots & \\ & & & \sqrt{\alpha} & 0 & \sqrt{\alpha} \\ & & & & \sqrt{\alpha} & \boxed{-g} \end{bmatrix}$$

Then

$$\hat{U} = SU \, , \tag{4.21}$$

where U is of the form (4.4).

3. Consider the following two quasi block Toeplitz matrices of order mn

$$A_* = \sum_{r=0}^{p} B_r \times A_r \, , \qquad p \leq n-1 \tag{4.22}$$

and

$$\hat{A}_* = \sum_{r=0}^{p} \hat{B}_r \times A_r \, , \qquad p \leq n-1 \tag{4.23}$$

where A_r $(r=0,1,\ldots,p)$ are arbitrary $m \times m$ matrices. They are given explicitly by (4.24) for $n=7$, $p=3$ and by (4.25) for $n=7$, $p=2$, respectively.

$$A_* = \begin{bmatrix}
A_0 - fA_1 - (1-f^2)A_2 - f(1-f^2)A_3 & A_1 - fA_2 - (1-f^2)A_3 & A_2 - fA_3 & A_3 & & & \\
A_1 - fA_2 - (1-f^2)A_3 & A_0 - fA_3 & A_1 & A_2 & A_3 & & \\
A_2 - fA_3 & A_1 & A_0 & A_1 & A_2 & A_3 & \\
A_3 & A_2 & A_1 & A_0 & A_1 & A_2 & A_3 \\
 & A_3 & A_2 & A_1 & A_0 & A_1 & A_2 - gA_3 \\
 & & A_3 & A_2 & A_1 & A_0 - gA_3 & A_1 - gA_2 - (1-g^2)A_3 \\
 & & & A_3 & A_2 - gA_3 & A_1 - gA_2 - (1-g^2)A_3 & A_0 - gA_1 - (1-g^2)A_2 - g(1-g^2)A_3
\end{bmatrix} \tag{4.24}$$

$$\hat{A}_* = \left[\begin{array}{c|c|c|c|c|c|c}
A_0 - fA_1 - (\alpha - f^2)A_2 & A_1 - fA_2 & A_2 & & & & \\ \hline
\alpha(A_1 - fA_2) & A_0 & A_1 & A_2 & & & \\ \hline
\alpha^2 A_2 & \alpha A_1 & A_0 & A_1 & A_2 & & \\ \hline
 & \alpha^2 A_2 & \alpha A_1 & A_0 & A_1 & A_2 & \\ \hline
 & & \alpha^2 A_2 & \alpha A_1 & A_0 & A_1 & A_2 \\ \hline
 & & & \alpha^2 A_2 & \alpha A_1 & A_0 & A_1 - gA_2 \\ \hline
 & & & & \alpha^2 A_2 & \alpha(A_1 - gA_2) & A_0 - gA_1 - (\alpha - g^2)A_2
\end{array}\right] \tag{4.25}$$

$$|f|, |g| \leq 1$$

<u>THEOREM</u> <u>4.1</u>: Matrices A_* (4.22) and $\hat{A}_*$ (4.23) have the explicit block diagonal decompositions

$$A_* = U_* T_* U_*^T \tag{4.26}$$

and

$$\hat{A}_* = \hat{U}_* \hat{T}_* \hat{U}_*^{-1} \tag{4.27}$$

Here

$$U_* = U \times I_m \,, \qquad \hat{U}_* = \hat{U}_* = \hat{U} \times I_m = SU \times I_m \tag{4.28}$$

$$T_* = \sum_{r=0}^{p} M_r \times A_r = \lceil\, T_j \,\rfloor_{j=1}^{n} \tag{4.29}$$

$$\hat{T}_* = \sum_{r=0}^{p} \hat{M}_r \times A_r = \lceil \hat{T}_j \rfloor_{j=1}^{n} , \tag{4.30}$$

$$T_j = A_o + 2 \sum_{r=1}^{p} A_r \cos r\theta_j , \tag{4.31}$$

$$\hat{T}_j = A_o + 2 \sum_{r=1}^{p} A_r \alpha^{r/2} \cos r\theta_j , \tag{4.32}$$

THEOREM 4.2: If matrices A_r, $r=0,1,\dots,p$ commute, i.e., they can be presented in the form

$$A_r = VN_rV^{-1} , \; N_r = \lceil \nu_\tau^{(r)} \rfloor_{\tau=1}^{m} , \qquad r=0,1,\dots,p \tag{4.33}$$

then A_* (4.22) and $\hat{A}_*$ (4.23) possess the explicit spectral decompositions

$$A_* = W_* \Lambda_* W_*^T \tag{4.34}$$

and

$$\hat{A}_* = \hat{W}_* \hat{\Lambda}_* \hat{W}_*^{-1} , \tag{4.35}$$

where

$$W_* = U \times V , \qquad \hat{W}_* = \hat{U} \times V = SU \times V , \tag{4.36}$$

$$\Lambda_* = \lceil \lceil \lambda_{j\tau} \rfloor_{\tau=1}^{m} \rfloor_{j=1}^{n} , \qquad \hat{\Lambda}_* = \lceil \lceil \hat{\lambda}_{j\tau} \rfloor_{\tau=1}^{m} \rfloor_{j=1}^{n} , \tag{4.37}$$

$$\lambda_{j\tau} = \nu_\tau^{(o)} + 2 \sum_{r=1}^{p} \nu_\tau^{(r)} \cos r\theta_j , \quad \hat{\lambda}_{j\tau} = \nu_\tau^{(o)} + 2 \sum_{r=1}^{p} \nu_\tau^{(r)} \alpha^{r/2} \cos r\theta_j \tag{4.38}$$

Clearly, both theorems are particular cases of Theorem 2.4. Symmetric quasi block Toeplitz matrices occur in applications usually more often. Note that matrix A_* (4.22) becomes symmetric if all submatrices A_r are symmetric. Matrix $\hat{A}_*$ (4.23) will be symmetric if (i) all even blocks $A_0, A_2, \ldots$ be symmetric, (ii) all odd blocks $A_1, A_3, \ldots$ be skew-symmetric, and (iii) $\alpha = -1$, $f = g = 0$.

4. Consider Hermitian quasi block Toeplitz matrices. With no loss in generality consider tri-block diagonal matrix

$$\underline{A}_* = \begin{bmatrix} A_0' & A_1 & & & & \\ A_1^H & A_0 & A_1 & & & \\ & A_1^H & A_0 & A_1 & & \\ & & \cdots & \cdots & \cdots & \\ & & & A_1^H & A_0 & A_1 \\ & & & & A_1^H & A_0'' \end{bmatrix}_{mn} \tag{4.39}$$

THEOREM 4.3: If all blocks of A_* (4.39) commute, i.e., if they can be expressed as

$$\begin{aligned} A_0 &= UMU^H, \quad & A_0' &= UM'U^H, \quad & A_0'' &= UM''U^H, \\ A_1 &= UNU^H, \quad & A_1^H &= U\bar{N}U^H, \end{aligned} \tag{4.40}$$

where

$$U = [u_{\sigma\tau}]_{\sigma,\tau=1}^{m}, \quad M = \lceil \mu_\tau \rfloor_{\tau=1}^{m}, \quad N = \lceil \nu_\tau \rfloor_{\tau=1}^{m}, \tag{4.41}$$

then there exists the following explicit block diagonal decomposition of A_* (4.39):

$$A_* = X_* \tilde{T}_* X_*^{-1} , \tag{4.42}$$

where

$$X_* = [[\delta_{ij} (\frac{|\nu_\tau|}{\nu_\tau})^i u_{\sigma\tau}]_{\sigma,j=1}^{m,n}]_{i,\tau=1}^{n,m} , \tag{4.43}$$

$$\tilde{T}_* = \lceil \tilde{T}_\tau \rfloor_{\tau=1}^{m} , \tag{4.44}$$

$$\tilde{T}_\tau = \begin{bmatrix} \mu'_\tau & |\nu_\tau| & & & & \\ |\nu_\tau| & \mu_\tau & |\nu_\tau| & & & \\ & |\nu_\tau| & \mu_\tau & |\nu_\tau| & & \\ & & \cdot & \cdot \; \cdot \; \cdot & \cdot & \\ & & & |\nu_\tau| & \mu_\tau & |\nu_\tau| \\ & & & & |\nu_\tau| & \mu''_\tau \end{bmatrix}_n \tag{4.45}$$

This theorem is a particular case of Theorem 1.1. Clearly, if

$$|f_\tau|, |g_\tau| \le 1 , \qquad \tau=1,\ldots,m \tag{4.46}$$

where

$$f_\tau = \frac{\mu_\tau - \mu_\tau'}{|\nu_\tau|}\ , \qquad g_\tau = \frac{\mu_\tau - \mu_\tau''}{|\nu_\tau|}\ , \qquad \tau=1,\ldots,m \tag{4.47}$$

then A_* (4.39) has the explicit spectral decomposition (Theorem 1.3). It will be also true if $|f|,|g| \leq 1$, where

$$A_o' = A_o - f(U|N|U^H)\ , \qquad A_o'' = A_o - g(U|N|U^H)\ , \qquad |N| = \lceil\ |\nu_\tau|\ \rfloor_{\tau=1}^{m} \tag{4.48}$$

5. FAST BLOCK ELIMINATION FOR SOLVING QUASI BLOCK TOEPLITZ SYSTEMS OF LINEAR EQUATIONS

1. Without loss of generality consider the following symmetric tri-block diagonal system of order mn with blocks of order m

$$\begin{bmatrix} A' & B & & & & \\ B^T & A & B & & & \\ & B^T & A & B & & \\ & & \cdots & & & \\ & & & B^T & A & B \\ & & & & B^T & A'' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdots \\ b_{n-1} \\ b_n \end{bmatrix} \qquad (5.1)$$

Matrix A_* contains only four different nonzero blocks A, A', A'', and B. All numerical methods [3] which can be applied to (5.1) ignore this property, moreover, they totally destroy it during elimination (block elimination). However, we can preserve and exploit this essential property if at each step of the forward pass we shall simultaneously eliminate all even subvectors $x_2, x_4, x_6, \ldots$. Then a new system, obtained at each step of such block elimination, will have the same block periodic configuration (5.1), but its order will be approximately half the previous one:

$$n_{k+1} = E(n_k/2) = E\left((n_o-1)/2^{k+1}\right) + 1 , \qquad n_o \equiv n , \qquad k=0,1,2,\ldots \quad (5.2)$$

E(x) is the greatest integer of x.

2. This idea is very attractive, however, it becomes efficient only if we shall accompany it with a special numerical procedure in computation of desirable blocks $A_{k+1}, A'_{k+1}, A''_{k+1}, B_{k+1}$ and subvectors b_j^{k+1}, $k=1,2,\ldots$ at each step. Let us compute them from the following equations

$$\left.\begin{aligned} A_{k+1} &= A_k + U + W \\ B_{k+1} &= V \\ A'_{k+1} &= A'_k + U \\ A''_{k+1} &= \begin{cases} A''_k + W & \text{for odd } n_k \\ A_k + \tilde{U} + W & \text{for even } n_k \end{cases} \end{aligned}\right\} \qquad (5.3)$$

$$\left.\begin{aligned} b_j^{k+1} &= b_{2j}^k + z_{2j} + y_{2j-2}, \qquad j=1,2,\ldots,n_{k+1}-1 \\ b_{n_{k+1}}^{k+1} &= \begin{cases} b_{n_k}^k + y_{n_k-1} & \text{for odd } n_k \\ b_{n_k-1}^k + \tilde{x}_{n_k} + y_{n_k-2} & \text{for even } n_k \end{cases} \end{aligned}\right\} \qquad (5.4)$$

where blocks U, V, W, $\tilde{U}$ and subvectors z_{2j}, $y_{2j-2}, \ldots$ are found by standard Gaussian elimination applied to the following augmented matrices

$$\left[\begin{array}{ccc|c} A_k & B_k^T & B_k & \ldots b_j^k \ldots \\ \hline B_k & 0 & 0 & \ldots 0 \ldots \\ \hline B_k^T & 0 & 0 & \ldots 0 \ldots \end{array}\right] \Longrightarrow \left[\begin{array}{c|cc|c} R_k & S_k & T_k & \ldots c_j^k \ldots \\ \hline 0 & U & V & \ldots z_j \ldots \\ \hline & V^T & W & \ldots y_j \ldots \end{array}\right] \tag{5.5}$$

and

$$\left[\begin{array}{cc|c} A''_k & B_k^T & b_{n_k}^k \\ \hline B_k & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{c|c|c} \tilde{R}_k & \tilde{S}_k & \tilde{c}_{n_k} \\ \hline 0 & \tilde{U} & \tilde{z}_{n_k} \end{array}\right] \tag{5.6}$$

The last transformation has to be done only if n_k is even. Certainly, only upper triangular parts of U, W and $\tilde{U}$ are computed. After p steps of such block elimination, where

$$p = E(\log_2 (n-1)) , \tag{5.7}$$

we obtain the system of order $mn_p = 2m$

$$\begin{bmatrix} A_p' & B_p \\ B_p^T & A_p'' \end{bmatrix} \begin{bmatrix} x_1^p \\ x_2^p \end{bmatrix} = \begin{bmatrix} b_1^p \\ b_2^p \end{bmatrix} , \tag{5.8}$$

which does not contain repeated blocks and has to be solved by any standard method.

3. Back substitution also requires p steps. At the k^{th} step of the backward pass we compute subvectors $x_2^{p-k}, x_4^{p-k}, \ldots$ which have been eliminated at the $p-k^{th}$ step of the forward pass. Upper triangular matrices R_{p-k} (5.5) and $\tilde{R}_{p-k}$ (5.6) are used in this case:

$$R_{p-k}[x_2^{p-k}, x_4^{p-k}, \ldots, x_{t_{p-k}}^{p-k}] = [c_2^{p-k}, c_4^{p-k}, \ldots, c_{t_{p-k}}^{p-k}]$$

$$- [S_{p-k}T_{p-k}] \begin{bmatrix} x_1^{p-k+1} & x_2^{p-k+1} & & x_{n_{p-k+1}-1}^{p-k+1} \\ & , & ,\ldots, & \\ x_2^{p-k+1} & x_3^{p-k+1} & & x_{n_{p-k+1}}^{p-k+1} \end{bmatrix} , \tag{5.9}$$

where

$$t_{p-k} = \begin{cases} n_{p-k}-1 , & \text{if } n_{p-k} \text{ is odd} \\ n_{p-k}-2 , & \text{otherwise} \end{cases} \tag{5.10}$$

4. Clearly, each step of this procedure requires more computation than standard one, but a number of steps is very small. Therefore the efficiency of Fast Block Elimination will increase with $\alpha = n/m$ increase.

Compare this method with the One-Way Dissection Method (1WD) [3].* Let (N_1, M_1) be a numer of operations and storage requirement of the proposed method, respectively. The same quantities for the 1WD are (N_2^*, M_2^*) and (N_2^{**}, M_2^{**}), where N_2^* and M_2^{**} are optimal number of operations and optimal storage requirement, respectively. Then the efficiency of Fast Block Elimination can be characterized by the following four ratios as functions in m and n

$$R_N^* = \frac{N_2^*}{N_1}, \qquad R_M^* = \frac{M_2^*}{M_1}$$

and

$$R_N^{**} = \frac{N_2^{**}}{N_1}, \qquad R_M^{**} = \frac{M_2^{**}}{M_1}$$

Corresponding curves R(m,n) are depicted in Figures 1 and 2. From them it is concluded that the proposed method is preferable $(R > 1)$ when

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*The Nested Dissection Method (ND) is more sophisticated. However, it has approximately the same efficiency as the 1WD [3].

$$\alpha = n/m \gtrsim 1.3^* \tag{5.11}$$

5. The proposed method is also applicable to the eigenvalue problem. Suppose we use the Sturm's method whose procedure requires factorization, by Gaussian elimination, of the matrix $B_*(\lambda) = \lambda I - A_*$ for some particular value of λ and to compute the negative index of inertia σ^- (i.e., a number of negative Gaussian pivots of $B_*(\lambda)$ transformed to a triangular form). Then applying the Fast Block Elimination one can take advantage of repeated blocks in computation of σ^-.

*More precisely: if $n \leq 40$, $\alpha \geq 1.5$, if $n = 50\text{-}60$, $\alpha \geq 1.1\text{-}1.2$, if $n \geq 70$, $\alpha \geq 1$.

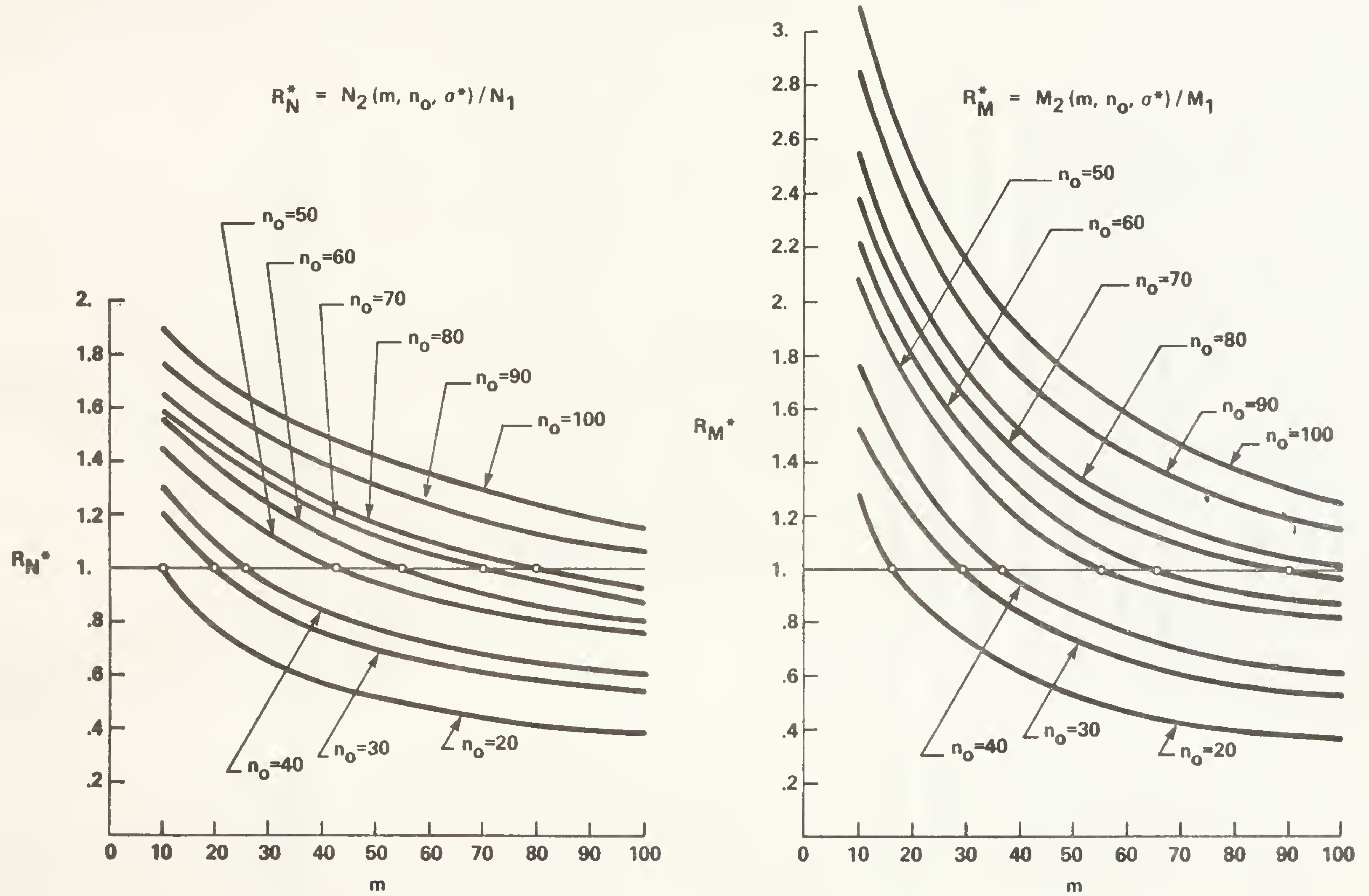

FIGURE 1 COMPARISON BETWEEN THE PROPOSED METHOD AND THE ONE-WAY DISSECTION METHOD. σ^* MINIMIZES THE OPERATION COUNT

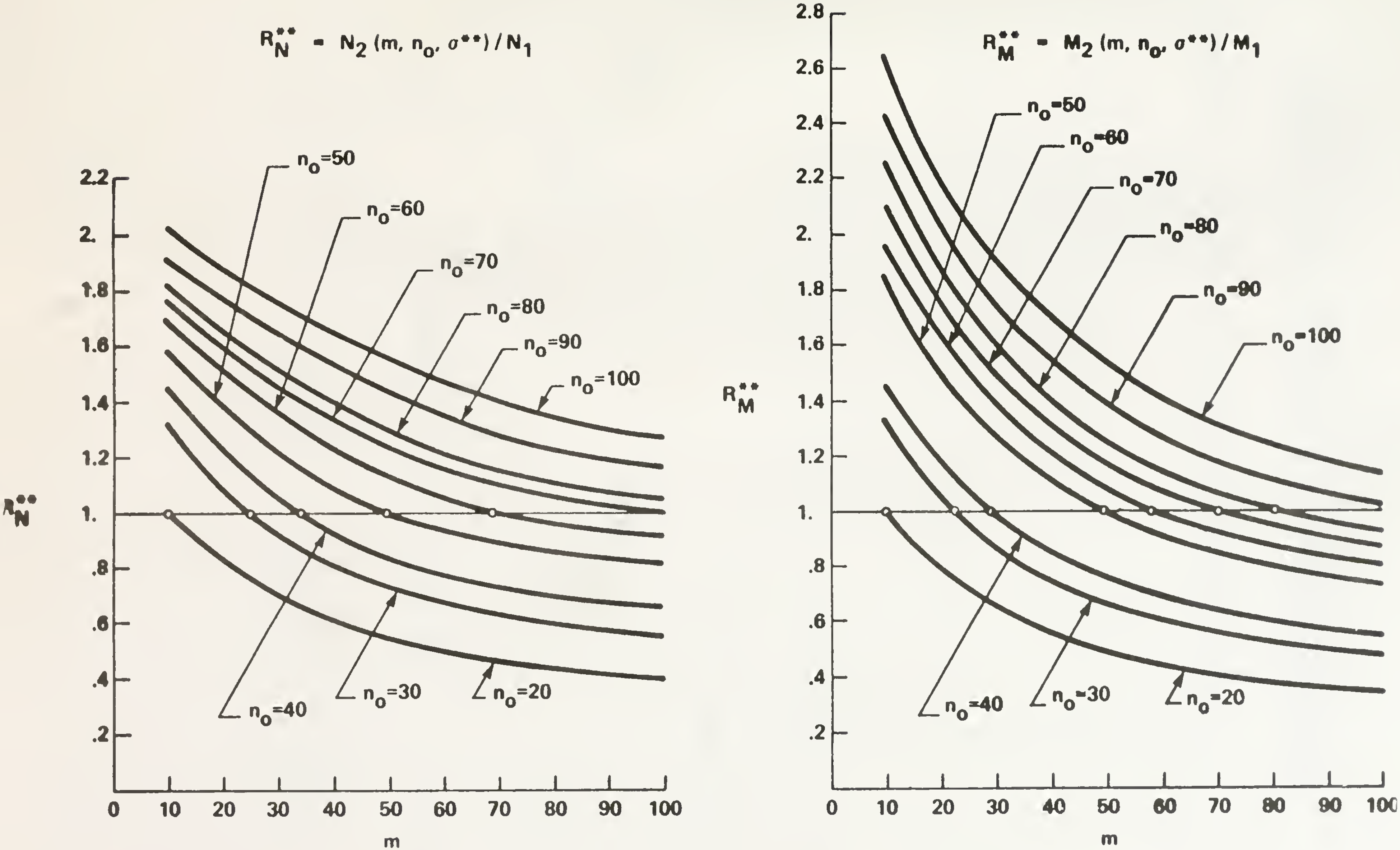

FIGURE 2 COMPARISON BETWEEN THE PROPOSED METHOD AND THE ONE-WAY DISSECTION METHOD. σ^{**} MINIMIZES STORAGE REQUIREMENTS

6. EXPLICIT ASYMPTOTIC SPECTRAL DECOMPOSITION OF FINITE DIFFERENCE BLOCK MATRIX CORRESPONDING TO BIHARMONIC OPERATOR OF A REGULAR RECTANGULAR MESH

1. The 13-point molecule for a biharmonic operator on a rectangle

$$\Delta_x \Delta_x = \frac{1}{(\Delta x)^4} \begin{array}{|c|c|c|c|c|} \hline & & r & & \\ \hline & t & q & t & \\ \hline 1 & s & p & s & 1 \\ \hline & t & q & t & \\ \hline & & r & & \\ \hline \end{array} \qquad (6.1)$$

with boundary conditions

$$u\big|_{edge} = 0$$

and

$$\frac{\partial u}{\partial n}\bigg|_{edge} = 0 \quad \text{or} \quad \frac{\partial^2 u}{\partial n^2}\bigg|_{edge} = 0$$

leads to the following five block diagonal quasi Toeplitz matrix A_* of order mn with blocks of order m:

$$
\underline{A}_* = \begin{bmatrix}
\boxed{A \pm I_m} & B & I_m & & & & \\
B & A & B & I_m & & & \\
I_m & B & A & B & I_m & & \\
& \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
& & I_m & B & A & B & I_m \\
& & & I_m & B & A & B \\
& & & & I_m & B & \boxed{A \pm I_m}
\end{bmatrix}, \tag{6.2}
$$

where

$$
A = \begin{bmatrix}
\boxed{p \pm r} & q & r & & & & \\
q & p & q & r & & & \\
r & q & p & q & r & & \\
& \cdot & \cdot & \cdot & \cdot & \cdot & \\
& r & q & p & q & r & \\
& & r & q & p & q & \\
& & & r & q & \boxed{p \pm r}
\end{bmatrix}, \quad B = \begin{bmatrix}
s & t & & & & \\
t & s & t & & & \\
& t & s & t & & \\
& & \cdot & \cdot & \cdot & \\
& & t & s & t & \\
& & & t & s & t \\
& & & & t & s
\end{bmatrix} \tag{6.3}
$$

$$\left.\begin{aligned} p &= 6 + 8\gamma^2 + 6\gamma^4 \\ q &= -4\gamma^2(1+\gamma^2) \\ r &= \gamma^4 \\ s &= -4(1+\gamma^2) \\ t &= 2\gamma^2 \end{aligned}\right\} \tag{6.4}$$

$$\gamma = \frac{\Delta x}{\Delta y} \qquad \Delta x = \frac{\ell_x}{n+1}, \qquad \Delta y = \frac{\ell_y}{m+1} \tag{6.5}$$

Matrix A_* (6.2) and all its blocks are symmetric. The boundary blocks of A_* (6.2) and boundary elements of A (6.3) are $A \pm I_m$ and $p \pm r$, respectively. Positive sign corresponds to $\partial u/\partial n = 0$ (clamped size), while negative sign to $\partial^2 u/\partial n^2 = 0$ (simply supported size). If boundary blocks of A_* are $A-I_m$ or if boundary elements of A (6.3) are p-r, matrix A_* can be explicitly block diagonalized. If both such conditions are satisfied (a simply supported rectangular plate), matrix A_* has the explicit spectral decomposition, which can be viewed as a discrete Navier solution. We shall consider a case with $A+I_m$ and p+r, i.e., a clamped rectangular plate.

2. In accordance with Lemma 4.1 matrix B (6.3) has the explicit spectral decomposition

$$B = VMV, \tag{6.6}$$

$$M = \lceil\ \mu_\tau \rfloor_{\tau=1}^{m}, \qquad \mu_\tau = s + 2t \cos\frac{\tau\pi}{m+1}, \tag{6.7}$$

$$V^{-1} = V = [v_{\sigma\tau}]_{\sigma,\tau=1}^{m} , \qquad v_{\sigma\tau} = \sqrt{\frac{2}{m+1}} \sin \frac{\sigma\tau\pi}{m+1} \tag{6.8}$$

<u>LEMMA</u> <u>6.1</u>: There exists the following explicit spectral decomposition of matrix A (6.3)

$$A = U\Lambda U^T , \tag{6.9}$$

$$\Lambda = [\ \lambda_\tau\ ]_{\tau=1}^{m} , \qquad \lambda_\tau = p + 2q \cos \theta_\tau + 2r \cos 2r\theta_\tau , \tag{6.10}$$

$$U = [u_{\sigma\tau}]_{\tau,\tau=1}^{m} , \qquad U^{-1} = U^T , \tag{6.11}$$

$$u_{\sigma\tau} = c_\tau[\cosh \sigma\psi_\tau - \cos \sigma\theta_\tau - \frac{\cosh (m+1)\psi_\tau - \cos (m+1)\theta_\tau}{\dfrac{\sinh (m+1)\psi_\tau}{\sinh \psi_\tau} - \dfrac{\sin (m+1)\theta_\tau}{\sin \theta_\tau}} (\frac{\sinh \sigma\psi_\tau}{\sinh \psi_\tau} - \frac{\sin \sigma\theta_\tau}{\sin \theta_\tau})] \tag{6.12}$$

with ψ_τ determined from

$$\cosh \psi_\tau = - \frac{q}{2r} - \cos \theta_\tau , \qquad \tau=1,\ldots,m \tag{6.13}$$

Arguments θ_τ are zeros of the following characteristic polynomial

$$\begin{aligned}\Delta_m(\theta) &= 2[1 - \cos (m+1)\theta \cos (m+1)\psi - \frac{\sin^2 \theta - \sinh^2 \psi}{\sin \theta \sinh \psi} \sin (m+1)\theta \sinh (m+1)\psi] \\ &= [\sinh \psi \tan \frac{m+1}{2} \theta - \sin \theta \tanh \frac{m+1}{2} \psi] [\sin \theta \tan \frac{m+1}{2} \theta + \text{snh } \psi \tanh \frac{m+1}{2} \psi]\end{aligned} \tag{6.14}$$

where ψ is defined by

$$\cosh\psi = -\frac{q}{2r} - \cos\theta \qquad (6.15)$$

PROOF: Present equation $(A-\lambda I_m)u = 0$ as the following finite difference system

$$ru_{\sigma-2} + qu_{\sigma-1} + (p-\lambda)u_\sigma + qu_{\sigma+1} + ru_{\sigma+1} = 0\ , \qquad (6.16)$$

$$\left.\begin{array}{l} u_{-1} - u_1 = 0 \\ u_o = 0 \\ u_{m+1} = 0 \\ u_{m+2} - u_m = 0 \end{array}\right\} \qquad (6.17)$$

Present u_σ as η^σ and denote $y = \eta + \eta^{-1}$. Then we obtain from (6.16) $ry^2 + qy + p - 2r - \lambda = 0$. Present also λ in the following form $\lambda = p + 2q\cos\theta + 2r\cos 2\theta$, then

$$y_1 = 2\cos\theta\ , \qquad y_2 = -\frac{q}{r} - 2\cos\theta = 2\cosh\psi\ ,$$

ψ is real, since $-q/r = 4(1+\gamma^{-2}) \geq 4$. Thus, $\eta_{1,2} = e^{+i\theta}$, $\eta_{3,4} = e^{\pm\psi}$ and

$$u_\sigma = c_1\cos\sigma\theta + c_2\sin\sigma\theta + c_3\cosh\sigma\psi + c_4\sinh\sigma\psi\ .$$

Substituting into (6.17), we obtain (6.12) and (6.14).

□

It is important to note that there are m distinct zeros θ_τ on $(0,\pi]$

$$\frac{\tau\pi}{m+1} < \theta_\tau < \frac{(\tau+.5)\pi}{m+1}\,, \qquad \tau=1,\ldots,m \tag{6.18}$$

Finally, we present θ_τ in the form

$$\theta_\tau = \frac{(\tau+\varepsilon_\tau)\pi}{m+1}\,, \qquad 0 < \varepsilon_\tau < .5\,, \qquad \tau=1,\ldots,m \tag{6.19}$$

3. <u>LEMMA</u> <u>6.2</u>:

$$A_* = I_n \times A + W_* N_* W_*^T\,, \tag{6.20}$$

where

$$W_* = V_* U_*\,, \tag{6.21}$$

$$V_* = I_n \times V\,, \qquad V\ (6.8)\,, \tag{6.22}$$

$$U_* = [U_{ij}]_{i,j=1}^n = [\ \lceil\ u_{ij}^{(\tau)}\ \rfloor_{\tau=1}^m]_{i,j=1}^n\,, \tag{6.23}$$

$$N_* = \lceil\ N_j\ \rfloor_{j=1}^n = \lceil\ \lceil\ \nu_j^{(\tau)}\ \rfloor_{\tau=1}^m\ \rfloor_{j=1}^n\,, \tag{6.24}$$

$$\nu_j^{(\tau)} = 2s\cos\theta_j^{(\tau)} + 4t\cos\frac{\tau\pi}{m+1}\cos\theta_j^{(\tau)} + 2\cos 2\theta_j^{(\tau)} \tag{6.25}$$

$$u_{ij}^{(\tau)} = c_j^{(\tau)} \Big[\cosh i\psi_j^{(\tau)} - \cos i\theta_j^{(\tau)}$$

$$- \frac{\cosh (n+1)\psi_j^{(\tau)} - \cos (n+1)\theta_j^{(\tau)}}{\dfrac{\sinh (n+1)\psi_j^{(\tau)}}{\sinh \psi_j^{(\tau)}} - \dfrac{\sin (n+1)\theta_j^{(\tau)}}{\sin \theta_j^{(\tau)}}} \Big(\frac{\sinh i\psi_j^{(\tau)}}{\sinh \psi_j^{(\tau)}} - \frac{\sin i\theta_j^{(\tau)}}{\sin \theta_j^{(\tau)}}\Big)\Big] \quad (6.26)$$

where $\psi_j^{(\tau)}$ are found from

$$\cosh \psi_j^{(\tau)} = -\frac{1}{2} s - t \cos \frac{\tau\pi}{m+1} - \cos \theta_j^{(\tau)} \quad (6.27)$$

Arguments $\theta_j^{(\tau)}$, j=1,...,n, are zeros of the polynomial

$$\Delta_n^{(\tau)}(\theta) = 2\Big[1 - \cos (n+1)\theta \cosh (n+1)\psi^{(\tau)}\Big]$$

$$- \frac{\sin^2 \theta - \sinh^2 \psi^{(\tau)}}{\sin \theta \sinh \psi^{(\tau)}} \sin (n+1)\theta \sinh (n+1)\psi^{(\tau)}$$

$$= \Big[\sinh \psi^{(\tau)} \tan \frac{n+1}{2} \theta - \sin \theta \tanh \frac{n+1}{2} \psi^{(\tau)}\Big]$$

$$\times \Big[\sin \theta \tan \frac{n+1}{2} \theta + \sinh \psi^{(\tau)} \tanh \frac{n+1}{2} \psi^{(\tau)}\Big] , \quad (6.28)$$

where $\psi^{(\tau)}$ is defined by

$$\cosh \psi^{(\tau)} = -\frac{1}{2} s - t \cos \frac{\tau\pi}{m+1} - \cos \theta \quad (6.29)$$

To prove this lemma, present block matrix A_* (6.2) in the following form

$$A_* = I_n \times A + (I_n \times V) C_* (I_n \times V) , \tag{6.30}$$

where C_* is a block matrix with diagonal blocks

$$C_* = \begin{bmatrix} I_m & M & I_m & & & & \\ M & 0 & M & I_m & & & \\ I_m & M & 0 & M & I_m & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & I_m & M & 0 & M & I_m \\ & & & I_m & M & 0 & M \\ & & & & I_m & M & I_m \end{bmatrix} \tag{6.31}$$

Then matrix $P_*^T C_* P_*$ is block diagonal with $n \times n$ blocks $\tilde{C}_\tau$

$$P_*^T C_* P_* = \lceil \tilde{C}_\tau \rfloor_{\tau=1}^{m} , \tag{6.32}$$

$$\tilde{C}_\tau = \begin{bmatrix} 1 & \mu_\tau & 1 & & & & \\ \mu_\tau & 0 & \mu_\tau & 1 & & & \\ 1 & \mu_\tau & 0 & \mu_\tau & 1 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & 1 & \mu_\tau & 0 & \mu_\tau & 1 \\ & & & 1 & \mu_\tau & 0 & \mu_\tau \\ & & & & 1 & \mu_\tau & 1 \end{bmatrix} , \quad \tau = 1, \ldots, m \tag{6.33}$$

Blocks $\tilde{C}_\tau$ have the same configuration as A (6.3), therefore Lemma 6.1 can be applied:

$$\tilde{C}_\tau = \tilde{U}_\tau \tilde{N}_\tau \tilde{U}_\tau^T , \qquad \tau=1,\ldots,m \tag{6.34}$$

$$\tilde{U}_\tau = [u_{ij}^{(\tau)}]_{i,j=1}^n , \qquad \tilde{N}_\tau = \lceil \nu_j^{(\tau)} \rfloor_{j=1}^n \tag{6.35}$$

Elements $\nu_j^{(\tau)}$ and $u_{ij}^{(\tau)}$ are determined by (6.25) thru (6.29). Clearly, the explicit spectral decomposition of C_* (6.31) is

$$\begin{aligned} C_* &= P_* \lceil \tilde{C}_\tau \rfloor P_*^T \\ &= (P_* \lceil \tilde{U}_\tau \rfloor_{\tau=1}^m P_*^T)(P_* \lceil \tilde{N}_\tau \rfloor_{\tau=1}^m P_*^T)(P_* \lceil \tilde{U}_\tau^T \rfloor_{\tau=1}^m P_*^T) \\ &= U_* N_* U_*^T , \end{aligned} \tag{6.36}$$

where U_* and N_* are given by (6.23) and (6.24), respectively. Finaly, substituting (6.36) into (6.30) we obtain (6.20)-(6.22). This completes the proof.

<u>REMARK</u>: Each characteristic polynomial $\Delta_n^{(\tau)}(\theta)$, $\tau=1,\ldots,m$ has n distinct zeros $\theta_j^{(\tau)}$, $j=1,\ldots,n$ on $(0,\pi]$

$$\frac{j\pi}{n+1} < \theta_j^{(\tau)} < \frac{(j+.5)\pi}{n+1} , \qquad j=1,\ldots,n \tag{6.37}$$

We present them in the form

$$\theta_j^{(\tau)} = \frac{(j+\varepsilon_j^{(\tau)})\pi}{n+1}, \qquad 0 < \varepsilon_j^{(\tau)} < .5, \qquad j=1,\ldots,n;\ \tau=1,\ldots,m \tag{6.38}$$

It is important to note that there exists the limit

$$\varepsilon_j = \lim_{n\to\infty} \varepsilon_j^{(\tau)}, \tag{6.39}$$

which is the same for all $\tau=1,\ldots,m$.

4. <u>LEMMA</u> <u>6.3</u>: Block matrix A_* (6.2) can be presented in the following form

$$A_* = X_* H_* X_*^T + O(\gamma^4), \tag{6.40}$$

where H_* is block diagonal

$$H_* = \lceil H_j \rfloor_{j=1}^{n}, \qquad H_j = A + VN_jV \tag{6.41}$$

and

$$X_* = V_* U_* V_* \tag{6.42}$$

<u>PROOF</u>: Introduce n^2 matrices X_{ij} similar to U_{ij}

$$X_{ij} = VU_{ij}V = [x_{\sigma\tau}^{(ij)}]_{\sigma,\tau=1}^{m} \tag{6.43}$$

Their elements

$$x_{\sigma\tau}^{(ij)} = \frac{2}{m+1} \sum_{\alpha=1}^{m} u_{ij}^{(\alpha)} \sin \frac{\sigma\alpha\pi}{m+1} \sin \frac{\alpha\tau\pi}{m+1} \tag{6.44}$$

possess periodic properties which leads to

$$[A, X_{ij}] = 2r \left[\begin{array}{c|c|c} 0 & \cdots\ x_{1\tau}^{(ij)}\ \cdots & 0 \\ \hline \begin{matrix}\vdots \\ -x_{\sigma 1}^{(ij)} \\ \vdots\end{matrix} & \mathbb{O}_{m-1} & \begin{matrix}\vdots \\ -x_{\sigma m}^{(ij)} \\ \vdots\end{matrix} \\ \hline 0 & \cdots\ x_{m\tau}^{(ij)}\ \cdots & 0 \end{array}\right] \tag{6.45}$$

$$i,j = 1,\ldots,n$$

Since $r = \gamma^4$ and $x_{\sigma\tau}^{(ij)} = O(1)$,

$$AX_{ij} = X_{ij}A + O(\gamma^4)\ , \qquad i,j=1,\ldots,n$$

and

$$AVU_{ij} = VU_{ij}(VAV) + O(\gamma^4)\ , \qquad i,j=1,\ldots,n$$

Clearly, $VU_{ij} = W_{ij}$ - blocks of W_* (6.21). Thus

$$AW_{ij} = W_{ij}(VAV) + O(\gamma^4) , \qquad i,j=1,\dots,n \tag{6.46}$$

or

$$(I_n \times A)W_* = W_*(I_n \times VAV) + O(\gamma^4)$$

and

$$I_n \times A = W_*(I_n \times VAV)W_*^T + O(\gamma^4) \tag{6.47}$$

Introduce this equation into (6.20)

$$A_* = W_*[I_n \times VAV + N_*]W_*^T + O(\gamma^4)$$

Obviously we can write it in the form of (6.40).

□

5. Elements $\nu_j^{(\tau)}$ (6.25) of blocks N_j (6.24) can be written as

$$\nu_j^{(\tau)} = 2\mu_\tau \cos \frac{(j+\varepsilon_j^{(\tau)})\pi}{n+1} + 2 \cos \frac{2(j+\varepsilon_j^{(\tau)})\pi}{n+1} , \qquad j=1,\dots,n;\ \tau=1,\dots,m$$

Therefore according to (6.39)

$$\lim_{n\to\infty} \nu_j^{(\tau)} = 2\mu_\tau \cos \frac{(j+\varepsilon_j)\pi}{n+1} + 2 \cos \frac{2(j+\varepsilon_j)\pi}{n+1} = 2\mu_\tau \cos \theta_j + 2 \cos 2\theta_j ,$$

$$j=1,2,\dots;\ \tau=1,\dots,m$$

and

$$\lim_{n\to\infty} N_j = 2M \cos \theta_j + 2I_m \cos 2\theta_j, \quad j=1,2,\ldots \qquad (6.48)^*$$

Since $H_j = A + VN_jV$ (6.41), one can write, using (6.6),

$$\lim_{n\to\infty} H_j = \lim_{n\to\infty} (A + 2B \cos \theta_j + 2I_m \cos 2\theta_j), \quad j=1,2,\ldots \qquad (6.49)$$

This means that $\lim_{n\to\infty} H_j$, $j=1,2,\ldots$ have the same configurations as A (6.3)

$$\lim_{n\to\infty} H_j = \lim_{n\to\infty} \begin{bmatrix} h_{oj}+r & h_{1j} & r & & & & \\ h_{1j} & h_{oj} & h_{1j} & r & & & \\ r & h_{1j} & h_{oj} & h_{1j} & r & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & r & h_{1j} & h_{oj} & h_{1j} & r & \\ & & r & h_{1j} & h_{oj} & h_{1j} & \\ & & & r & h_{1j} & h_{oj}+r \end{bmatrix}_m, \qquad (6.50)$$

$j=1,2,\ldots$

*Note $\lim_{n\to\infty} \sum_{j=1}^{n} N_j = 2I_m$.

where

$$h_{oj} = p + 2s \cos \theta_j + 2 \cos 2\theta_j$$
$$h_{1j} = q + 2t \cos \theta_j \tag{6.51}$$

LEMMA 6.4:

$$\lim_{n\to\infty} (H_j - U_j \Lambda_j U_j^T) = 0_m \,, \qquad j=1,2,\ldots \tag{6.52}$$

Here

$$U_j = \left[u_{j\sigma\tau} \right]_{\sigma,\tau=1}^{m} \,, \text{ (see (6.11))} \,,$$

$$\Lambda_j = \lceil \lambda_{j\tau} \rfloor_{\tau=1}^{m} \,,$$

$$\lambda_{j\tau} = p + 2s \cos \theta_j + 2 \cos 2\theta_j + 2q \cos \theta_\tau^{(j)} + 4t \cos \theta_j \cos \theta_\tau^{(j)} + 2r \cos 2$$

$$j=1,2,\ldots;\ \tau=1,\ldots,m \tag{6.53}$$

Finally, we obtain the following result.

THEOREM 6.1:

$$\lim_{n\to\infty} \left[A_* - Z_* \Lambda_* Z_* \right] = O(\gamma^4) \tag{6.54}$$

Here

$$Z_* = X_*(I_n \times U_j)\ , \qquad Z_{ij} = VU_{ij}VU_j \tag{6.55}$$

$$\Lambda_* = \lceil \lceil \lambda_{j\tau} \rfloor_{\tau=1}^{m} \rfloor_{j=1}^{n\to\infty}$$

Substituting (6.4) into (6.53), one can write

$$\lambda_{j\tau} = 16\ [\sin^2 \frac{\theta_j}{2} + \gamma^2 \sin^2 \frac{\theta_\tau^{(j)}}{2}]\ , \qquad \tau=1,2,\ldots,m;\ j=1,2,\ldots \tag{6.56}$$

6. THEOREM 6.2: Natural frequencies of a clamped rectangular plate have the following explicit expression

$$\phi_{j\tau} = \pi^2[\frac{(j+\varepsilon_j)^2}{\ell_x^2} + \frac{(\tau+\varepsilon_\tau)^2}{\ell_y^2}] \sqrt{\frac{D}{m_o}}\ , \qquad j,\tau=1,2,\ldots \tag{6.57}$$

Here D and m_o are the rigidity of a thin plate and mass per unit area, respectively,

$$\varepsilon_j = \frac{n+1}{\pi}\ \theta_j - j\ , \qquad \varepsilon_\tau = \frac{m+1}{\pi}\ \theta_\tau - \tau\ , \tag{6.58}$$

θ_j and θ_τ are zeros of $\Delta_n^{(\tau)}(\theta)$ and $\Delta_m(\theta)$, respectively. They are computed for large $m,n \geq 100$. Correction terms $\varepsilon_j, \varepsilon_\tau$ satisfy inequalities

$$0 < \varepsilon_j, \varepsilon_\tau < .5 \tag{6.59}$$

In fact,

$$\phi_{j\tau} = \lim_{n,m\to\infty} \frac{1}{(\Delta x)^2} \lambda_{jt}^{1/2} \sqrt{\frac{D}{m_o}}$$

$$= \left[\lim_{n\to\infty} \frac{4}{(\Delta x)^2} \sin^2 \frac{\theta_j}{2} + \lim_{m\to\infty} \frac{4}{(\Delta y)^2} \sin^2 \frac{\theta_\tau^{(j)}}{2}\right] \sqrt{\frac{D}{m_o}}$$

$$= \left[\lim_{n\to\infty} \frac{\pi^2(j+\varepsilon_j)^2}{(n+1)^2(\Delta x)^2} + \lim_{m\to\infty} \frac{\pi^2(\tau+\varepsilon_\tau)^2}{(m+1)^2(\Delta y)^2}\right] \sqrt{\frac{D}{m_o}}$$

$$= \pi^2\left[\frac{(j+\varepsilon_j)^2}{\ell_x^2} + \frac{(\tau+\varepsilon_\tau)^2}{\ell_y^2}\right] \sqrt{\frac{D}{m_o}}, \qquad j,\tau=1,2,\ldots$$

Note that eq. (6.57) generalizes the Navier formula ($\varepsilon_j = \varepsilon_\tau = 0$) for a simply supported rectangular plate (see Table).

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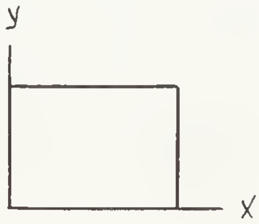
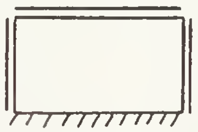
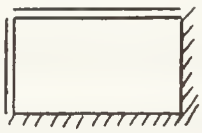
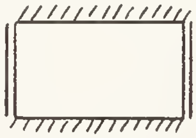
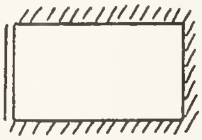
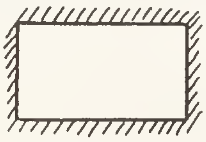
Table

#		ε_j, ε_τ
1		$\varepsilon_j = 0$ $\varepsilon_\tau = 0$
2		$0 < \varepsilon_j < .25$ $\varepsilon_\tau = 0$
3		$0 < \varepsilon_j < .25$ $0 < \varepsilon_\tau < .25$
4		$0 < \varepsilon_j < .5$ $\varepsilon_\tau = 0$
5		$0 < \varepsilon_j < .5$ $0 < \varepsilon_\tau < .25$
6		$0 < \varepsilon_j < .5$ $0 < \varepsilon_\tau < .5$

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NOTES

NOTES

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Dinkevich, Solomon
Explicit block diagonal
decomposition of block
matrices corresponding to

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